# Ground state structure of random magnets

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Using exact optimization methods, we find all of the ground states of ($\pm h$) random-field Ising magnets (RFIM) and of dilute antiferromagnets in a field (DAFF). The degenerate ground states are usually composed of isolated clusters (two-level systems) embedded in a frozen background. We calculate the paramagnetic response (sublattice response) and the ground state entropy for the RFIM (DAFF) due to these clusters. In both two and three dimensions there is a broad regime in which these quantities are strictly positive, even at irrational values of $h/J$ ($J$ is the exchange constant). [S1063-651X(98)05610-4]

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Disordered magnets [1] provide a paradigm for disordered systems in general, and they continue to be intensively analyzed by a variety of methods. However, due to metastability, conventional (e.g., Monte Carlo) analysis [2] of the equilibrium domain structure of random magnets is often unreliable, especially at low temperatures. Since the ground state behavior is an important indicator of the low temperature behavior of most random magnets [3], exact methods for ground state analysis are desirable. Fortunately, the *true* ground states of random magnets can often be found using optimization methods.

The relation between optimization and random magnets was pointed out some time ago [4,5]. However, extensive use of these methods is more recent, partially due to the availability of more efficient algorithms. An exact optimization procedure to find the random-field ground state was implemented by Ogielski [5]. More extensive analyses on larger system sizes have been published recently [6,7]. These methods have also been extended to the analysis of the ground state degeneracy of random magnets [8,9]. Here we present, using a new algorithm, a more precise analysis of the ground state degeneracy and its consequences in the random-field Ising model (RFIM) and the dilute antiferromagnet in a field (DAFF) in dimensions $d=2$ and 3 (square and cubic lattices). We concentrate on the following three aspects of these degenerate random magnets: (1) The degenerate domain structure of RFIM ground states (e.g., Fig.1). (2) The order parameter which couples to the ground state degeneracy. (3) The ground state entropy, and in particular the physical origin of its continuous and discontinuous parts.

We consider the RFIM with a binary random field,

$$H_{\mathrm{RFIM}}=-J\sum_{\langle ij\rangle}\sigma_i\sigma_j-\sum_i h_i\sigma_i, \tag{1}$$

where $h_i=\pm h$, with $h$ and $J$ positive, and the plus and minus random fields occur with equal probability. We also analyze the DAFF,

$$H_{\mathrm{DAFF}}=\sum_{\langle ij\rangle} Jx_ix_j\sigma_i\sigma_j-h\sum_i x_i\sigma_i, \tag{2}$$

where $p_i=p\,\delta(x_i-1)+(1-p)\,\delta(x_i)$ is the probability that a site is present. In both cases, we analyze the ground state properties as a function of the ratio $H=h/J$ on square and cubic lattices. In the DAFF case there is the additional parameter $p$, which we fix at $p=0.9$.

The ground state critical behavior of random field magnets is still not completely understood [3,6,10]. At small $H$, large ferromagnetic domains are favored, while at large $H>H_*$, the spins freeze along the directions of the local field

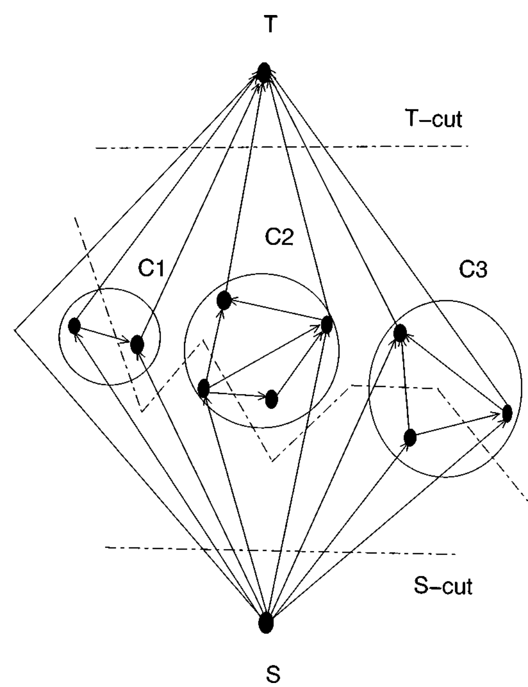


FIG. 1. Typical supergraph as obtained using the algorithm. $S$ is the set of spins frozen up, and $T$ the set of spins frozen down. $C_1$, $C_2$, and $C_3$ are independent clusters; they are made of subclusters that are not independent of each other. The $S$-cut is the ground state with all the independent clusters down, and the $T$-cut is the one with all of them up. We also show a directed cut (ground state) in which part of each independent cluster is up or down.

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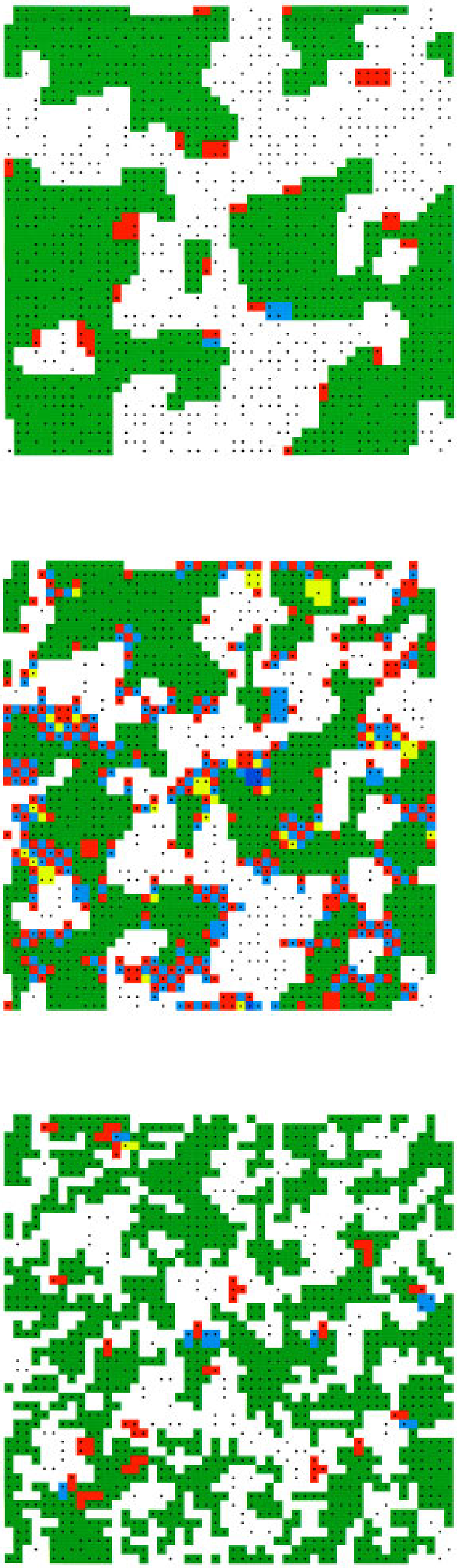

FIG. 2. (Color) Typical ground states of the $d=2$ random field ising model [Eq. (1)] with $H=h/J=3/2$ (top), $H=2$ (middle), and $H=\frac{5}{2}$ (bottom) (system size $50\times 50$). Green indicates an up spin, white a down spin, and the other colors indicate spin clusters that can be flipped up or down (but not all independently of each other) without changing the ground state energy. Black dots indicate the sites where the random field favors the up-spin orientation.

$h_i$. In two dimensions there is no spontaneous magnetization for any finite $H(H_c^{2D}=0)$, though there is a rapidly growing ferromagnetic domain size $l \sim \exp(1/H^2)$, which can masquerade as a phase transition at $H \sim 1$. In three dimensions, there is a spontaneously magnetized state at small $H < H_c$. Although the field theory analysis and early simulations suggested a continuous behavior in magnetization $m(H)$ as $H \to H_c^-$, precise numerical work using exact optimization methods suggests a large jump in $m$ at $H_c$ (for the $(\pm h)$ random field case $H_c^{3D}=2.21\pm 0.01$ and $\Delta m \sim 0.8$) [6]. The DAFF was introduced as a possible experimental realization of the RFIM [11], and an extensive literature has developed from this observation [12]. The DAFF is an antiferromagnet at small $H$, and at large $H > H_*$ all spins are polarized in the field direction. The DAFF *sublattice (staggered) magnetization* is believed to be qualitatively similar to that of the *magnetization* of the RFIM. Our calculations are for the ground state degeneracy in the nontrivial regime $0 < H < H_*$, where $H_* = 2D$ ($d$ is dimension) is the field amplitude beyond which *all* spins follow the local field direction and the ground state is nondegenerate (for both RFIM and DAFF).

The ground state degeneracy of the RFIM has been intensively studied in one dimension [13]. There has also been an analysis on Cayley trees [14], and an interesting analysis on the square lattice [15]. The latter paper did not have the advantage of exact optimization methods, and missed some of the key features of the ground state degeneracy. Although the infinite range model misses entirely the degeneracy we find here [14], the one-dimensional and Cayley tree models have several qualitative similarities with our results. More recently, Hartmann [9] presented a low precision calculation of the ground state degeneracy of random magnets, though the physics we elucidate here was not discussed by him.

In order to find the ground state of RFIM and DAFF, we use the mapping of these problems to a flow problem in combinatorial optimization (so called min-cut max-flow) [16]. This algorithm also gives the exact minimal energy surface in random networks [17,18]. This has been known for some time [5]; however, improved algorithms (push-relabel with global updates [19]) now allow optimization of $100^3$ lattices in a few minutes on a high end workstation. Our method relies on the concept of *residual graph* introduced by the network flow algorithms [19]. The full residual graph of the equivalent network flow problem holds the whole information about the ground state structure. A naive search of the ground states — which is equivalent to finding the domains that can be flipped without altering the energy (or the max-flow in the network flow terminology) — is exponential. Instead we generate a supergraph which shows how the domains are related to each other, i.e., which domains can be flipped independently. It turns out that many of the domains are independent, and the exponential search is reduced to the few remaining dependent domains, and we search over these remaining domains. We show that the problem is equivalent to finding all the directed cuts in a directed graph with single arcs and no cycles. A typical supergraph is shown in Fig. 1. It is easy to see how the structure of the supergraph makes such a search effective, as it reduces it to searches over much smaller independent graphs. Details of the method will be presented elsewhere [20].

A typical ground state domain structure of the two-dimensional $\pm h$ RFIM is presented in Fig. 2. Here green domains are fixed in the direction of the positive fields (dots), while white domains are fixed in the opposite direction. Domains of any other color can be flipped without changing the ground state energy (note that not all can be flipped independently, but the dependent domains are organized in clusters that can also be flipped as a whole). These domains produce a finite ground state entropy. Surprisingly, domains that can be flipped in the ground state exist *even for irrational* $H = h/J$. Note that these domains occur at the *interfaces* between the up-spin and down-spin domains of the RFIM ground state. The degenerate clusters at irrational $H$ have *zero field energy and the same exchange energy* in both the up and down states of the cluster. For the RFIM on the square lattice, the lowest order degenerate clusters of this sort are indicated in the inset to Fig. 3. The number of these clusters [or two-level-systems (TLS's)], $n_{\mathrm{TLS}}$, can be estimated using simple arguments:

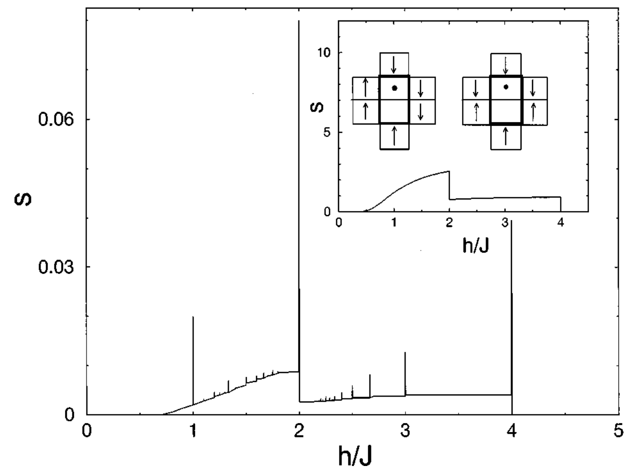


FIG. 3. The ground state entropy of the two-dimensional random field Ising model (RFIM) as a function of $h/J = H$. The inset shows the smallest two-level-systems (TLS's) at irrational $H$ and an estimate of the entropy [from Eq. (3)] produced by them. ''+'' indicates an up spin, ''−'' indicates a down spin, and a dot indicates where the local random field favors the ''+'' spin direction. The system sizes used were from $10\times 10$ to $130\times 130$ and the entropy was found as the slope of the line $\langle \ln D \rangle$ vs $N$, where $D$ is the degeneracy, $N$ is the system size (total number of spins), and the average is over the disorder (1000 samples were used).

$$n_{\mathrm{TLS}} \propto p_{\mathrm{TLS}} \frac{L^2}{l}, \tag{3}$$

where $L$ is the linear size of the system, $l \propto \exp[(1/H)^2]$ is the typical size of the ordered domains [3], and $L^2/l$ is the total length of interface between up- and down-spin domains in the system. $p_{\mathrm{TLS}}$ is the probability of occurrence of a TLS at a given interface site. $p_{\mathrm{TLS}} = p_n/4$, where $\frac{1}{4}$ is the probability of occurrence of the up-down pair of fields, and $p_n$ is the probability to have this pair surrounded in the ground state by frozen spins with the appropriate configurations. The entropy density is then $s \propto p_n \exp[-(1/H^2)]$ for $H < 4$ and 0 for $H > 4$. $p_n$ is discontinuous at $H = 2$, because the dominant TLS's for $H < 2$ are different than those for $H > 2$ (for example, there are twice as many spin configurations that lead

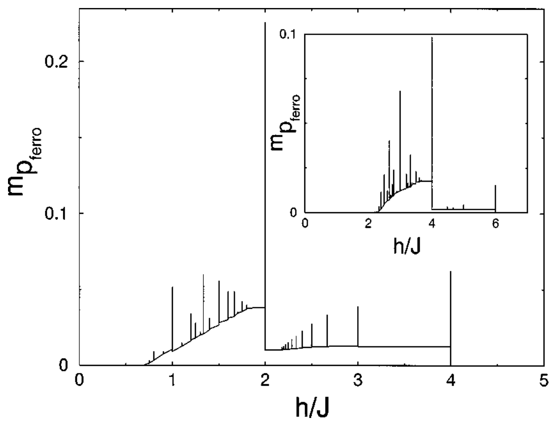


FIG. 4. The order parameter for the ground state paramagnetism ($m_{p_{\rm ferro}}$) of the RFIM on square and cubic (inset) lattices. The system size was 200×200 for the square and 40×40×40 to 60×60 ×60 for the cubic lattice (1000 samples were used).

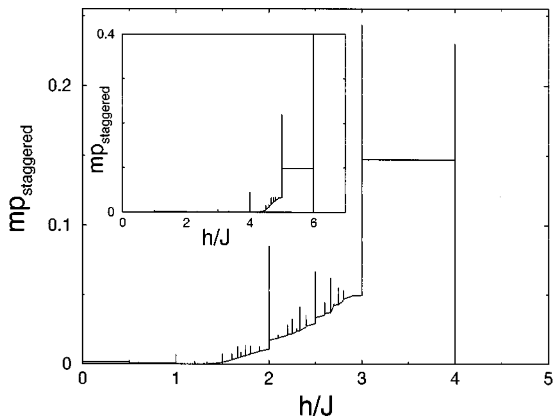


FIG. 5. The order parameter for the ground state paramagnetism ($m_{p_{\rm staggered}}$) of the DAFF on square and cubic (inset) lattices. The same system sizes and number of samples as for the RFIM were used.

to a TLS below than above $H=2$). If we take the observed jump ~3.3, then the above argument leads to the curve given in the inset to Fig. 3, which is very close in form to the continuous part of the entropy presented in Fig. 3.

The series of sharp peaks occurring at rational values of $H$ are due to additional degeneracy occurring when clusters have the same value for *the field energy plus the exchange energy* in either the up or down states. These peaks can only occur at *rational* values of $H$ and the cluster geometries which contribute at each rational are different. Naturally high order rationals correspond to complex clusters and have greatly reduced degeneracy. It has been suggested before [15] that for the two-dimensional (2D) RFIM, the highest degeneracies occur at rationals $H_n=2+2/n$ for $2<H<4$, with $n=1,2\ldots.$ Using our algorithm we checked this idea by considering $n=1,\ldots,11$, and all rationals with denominators 2, 3, 4, 5, and 6. We find that those with $H_n=2+2/n$ are indeed dominant [21] (see Fig. 3), and there is a similar sequence at fields $4+2/n$ in three dimensional (the zoology of the clusters leading to the dominant peaks is straightforward, though tedious to enumerate). In the regime $0<H<2$ the 2D RFIM ground state entropy has spikes at a large number of rationals (see Fig. 2). These features are quite similar to those found in one dimension (see Fig. 4 of Ref. [13]) and on Cayley trees (see Fig. 4 of Ref. [14]). We have also done a preliminary analysis of the 3D RFIM and the 2D and 3D DAFF ground states. In general we find that the RFIM and DAFF magnets in two dimensional and three dimensional are massively degenerate in the regime $H_c<H<H_*$ and that their ground state entropy is finite even at irrational $H$.

In the regime $H_c<H\leq H_*$, we can consider the ground state to be composed of a frozen background in which is embedded a set of largely noninteracting free *superspins* (corresponding to each independent cluster). The ground state of these magnets thus can be considered to contain a large number of magnetic two-level systems [22]. There is a *paramagnetic response* at low temperatures for both the RFIM and DAFF in this regime. The natural ground state order parameter for the paramagnetic response in the regime $H_c<H<H_*$ is the magnetization $m_{p_{\rm ferro}}$ for the RFIM and the staggered magnetization $m_{p_{\rm staggered}}$ for the DAFF. It is straightforward to calculate these order parameters using the exact optimization algorithm, either by applying an appropriate infinitesimal field or by polarizing all of the degenerate domains in a given orientation (we do the latter). The results for the RFIM are presented in Fig. 4 for both square and cubic (inset) lattices. It is seen that the basic features of the ground state degeneracy (Fig. 3) are reflected in the ground state paramagnetic magnetization. In three dimensions, the entropy remains zero at low $H$ (at least for H irrational), reflecting the ferromagnetic state for $H<H_c\sim 2.21$ [6]. For $H>H_*=6$ (in three dimensions), the spins are aligned with the random field, and the ground state is nondegenerate. Note that in addition to the paramagnetic magnetization, there is a spontaneous magnetization $m_{s_{\rm ferro}}$ for $H<H_c$. In experiments in which the ferromagnetic field is swept to produce a magnetization loop, the measured zero field magnetization is the sum, i.e., $m_{0_{\rm ferro}}=m_{s_{\rm ferro}}+m_{p_{\rm ferro}}$. Thus there is a finite equilibrium magnetization jump at zero temperature even for $H>H_c$. Of course $m_{p_{\rm ferro}}=0$ for $T>0$, but the effects of the ground state degeneracy should be reflected in magnetization anomalies and a Curie law in the susceptibility at low temperatures.

Calculations of the paramagnetic order parameter for the DAFF, $m_{p_{\rm staggered}}$, for square and cubic lattices is presented in Fig. 5. Qualitatively the situation is similar to that in the RFIM. There is a strong sublattice paramagnetic response for all $H_c<H<H_*$, with spikes at certain rational values. These figures are for a DAFF with dilution $p=0.9$, but only the details change as $p$ is varied. In the regime $H<H_c$ there is a spontaneous staggered magnetization, and low temperature measurements (such as neutron scattering and NMR) should be influenced by both the ''staggered paramagnetic'' response and the spontaneous staggered magnetization. We also note that the existence of the additional order parameter $m_{p_{\rm ferro}}$ in the case of the $\pm h$ RFIM suggests that the $\pm h$ RFIM and the Gaussian RFIM may not be in the same universality class.

We have described two developments in the analysis of random magnets. (i) Using optimization methods, it is possible to efficiently calculate the ground state structure of

RFIM and DAFF magnets (see Fig. 2). (ii) The $(\pm h)$ RFIM and the DAFF magnets have a spontaneously ordered state for $H < H_c$, a massively degenerate ground state in the regime $H_c \leqslant H \leqslant H_*$, and a nondegenerate ground state for $H > H_*$. In the degenerate regime there is a strictly positive paramagnetic response and ground state entropy even at irrational $H$, with additional degeneracies at rational $H$ (see Figs. 2–4).

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